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INVARIANT DIFFERENTIAL OPERATORS ON HERMITIAN SYMMETRIC SPACES AND THEIR EIGENVALUES

BY

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ABSTRACT

Let $\bar{D}$ be the invariant Cauchy Riemann operator and $\mathcal{M}_m = D^m\bar{D}^m$ the corresponding invariant Laplacians on a bounded symmetric domain. We calculate the eigenvalues of $\mathcal{M}_m$ on spherical functions. In particular we prove that for a symmetric domain of rank two the operators $\mathcal{M}_1, \mathcal{M}_3$ generate all invariant differential operators. We also find the eigenvalues of the generators introduced by Shimura.

Introduction

Let $\Omega = G/K$ be a Riemannian symmetric space and $\mathcal{D}_G(\Omega)$ the algebra of all G-invariant differential operators on Ω. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g} = Lie(G)$. It is now well-known (see e.g. [3]) that $\mathcal{D}_G(\Omega)$ is commutative and is isomorphic to the algebra of all Weyl group invariants in the space $\mathcal{P}(\mathfrak{a})$ of all polynomials on $\mathfrak{a}$. Here $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$. Thus there exists a system of $r = \dim(\mathfrak{a})$ operators that generates all $\mathcal{D}_G(\Omega)$. The Laplace-Beltrami operator can be chosen to be one of the generators. It is therefore a very interesting and natural question to find a geometric construction of a system of generators and calculate their eigenvalues. In the present paper we will consider the case of Hermitian symmetric spaces.

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In his paper [14] Shimura constructed a system of positive invariant differential operators that generates $\mathcal{D}_G(\Omega)$ on a classical Hermitian symmetric space Ω. His construction is purely algebraic. In the present paper we give a geometric construction of the Shimura operators. We study invariant differential operators $\mathcal{M}_m = D^m \bar{D}^m$ constructed via the invariant Cauchy-Riemann operator $\bar{D}$ on the space Ω and prove that $\mathcal{M}_m$ is a sum of the Shimura operators. We prove further that for rank two symmetric domains the (negative) operators $\mathcal{M}_1$ and $\mathcal{M}_3$ form also a system of generators, and thus prove a conjecture of Englis and Peetre [1] in that case. We find yet another system of generators using an idea of Rudin [10], whose eigenvalues can be calculated by using Berezin transform. We find then the eigenvalues of the generators $\mathcal{M}_1$ and $\mathcal{M}_3$ and of the generators of Shimura.

To explain our main results we take temporarily a Kähler manifold Ω with Kähler metric $h_{i\bar{j}}dz^i d\bar{z}^j$. Let W be a Hermitian vector bundle over Ω. The invariant Cauchy-Riemann operator $\bar{D}$ is introduced in [1], and defined by

$$\bar{D}(f^\alpha e_\alpha) = h^{\bar{j}i}\frac{\partial f^\alpha}{\partial \bar{z}^j}\partial_i \otimes e_\alpha,$$

where $f^\alpha e_\alpha$ is a section of the bundle W and e_α are local trivializing sections, namely they form a basis for the fibre space at each point. Thus $\bar{D}$ maps sections of the bundle W to sections of the bundle $T^{(1,0)} \otimes W$. Let $D = -D^*$ be its formal adjoint and $\mathcal{M}_m = D^m \bar{D}^m$ be the corresponding Laplacians on W. Thus $(-1)^m \mathcal{M}_m$ are positive operators. When Ω is the unit disk or a Riemann surface the operators $\bar{D}$ and $\mathcal{M}_m$ are introduced in [8]. It is further proved in [9] that the iterate $\bar{D}^m$ of $\bar{D}$ maps W into the subbundle $(\odot^m T^{(1,0)}) \otimes W$ of $(\otimes^m T^{(1,0)}) \otimes W$. Here $\odot^m$ stands for the symmetric tensor product. Consider now $\Omega = G/K$ a general irreducible bounded symmetric domain in a complex vector space V and identify the tangent space at any $z \in \Omega$ with V. The symmetric tensor product $\odot^m V$ is then decomposed under K into irreducible subspaces with signatures $\underline{\mathbf{m}} = (m_1, m_2, \ldots, m_r)$ with $|\underline{\mathbf{m}}| = m_1 + m_2 + \cdots + m_r = m$. For each $\underline{\mathbf{m}}$ let $P_{\underline{\mathbf{m}}}$ be the orthogonal projection onto the irreducible subspace. We prove in this paper for W a homogeneous vector bundle over Ω, the operator $\bar{D}$ is the Shimura operator E, and that the operators $D^m P_{\underline{\mathbf{m}}} \bar{D}^m$ are the Shimura Laplacians $\mathcal{M}_{\underline{\mathbf{m}}}$. Thus our Laplacian $\mathcal{M}_m = D^m \bar{D}^m$ is a sum of the Shimura Laplacians and we have given a geometric construction of the Shimura operators; see Proposition 3.2 below.

When W is the trivial line bundle on Ω, Shimura [14] proved that the operators $\mathcal{M}_{\underline{\mathbf{m}}}$, for $\underline{\mathbf{m}}$ being the fundamental representations $(1, 0, \ldots, 0)$, $(1, 1, 0, \ldots, 0)$,

..., $(1, 1, \ldots, 1)$, form a system of generators of the algebra $\mathcal{D}_G(\Omega)$. On the other hand, Englis and Peetre conjecture that the operators $\mathcal{M}_m$ generate the algebra $\mathcal{D}_G(\Omega)$. Our next goal is to find the eigenvalues of the generators in the Shimura system and $\mathcal{M}_m$, and thus to prove (or disprove) the Englis-Peetre conjecture. When Ω is of rank two, the Shimura system is then $(\mathcal{M}_{(1,0)}, \mathcal{M}_{(1,1)})$ and $\mathcal{M}_{(1,0)} = \mathcal{M}_1 = L$ is the Laplace-Beltrami operator. We are able to find the eigenvalues of the operator $\mathcal{M}_{(1,1)}$ and that of $\mathcal{M}_3$, and thus prove the Englis-Peetre conjecture in that case; see Theorem 6.5 and Theorem 5.6. We proceed to explain our method of calculating the eigenvalues.

The space of holomorphic polynomials $\mathcal{P}(V)$ on V, similarly to the symmetric tensor above, is decomposed under K into irreducible subspaces $\mathcal{P}^{\underline{\mathbf{m}}}$ of signatures $\underline{\mathbf{m}} = (m_1, m_2, \ldots, m_r)$, with multiplicity free. For each $\underline{\mathbf{m}}$ there corresponds a K-invariant polynomial $K_{\underline{\mathbf{m}}}(z, z)$ on V. Using an idea of Rudin [10] we construct a G-invariant differential operator $\mathcal{K}_{\underline{\mathbf{m}}}$ on Ω, which at the origin $z = 0$ is the differential operator $K_{\underline{\mathbf{m}}}(\partial, \bar{\partial})$. The eigenvalue of $\mathcal{K}_{\underline{\mathbf{m}}}$ on the spherical function $\phi_{\underline{\lambda}}$ is, roughly speaking, the coefficients in the expansion of $\phi_{\underline{\lambda}}(z)$ in terms of $K_{\underline{\mathbf{m}}}(z, z)$, which in turn is $K_{\underline{\mathbf{m}}}(\partial, \bar{\partial})\phi_{\underline{\lambda}}(0)$. Instead of performing this differentiation we consider the differentiation $\mathcal{K}_{\underline{\mathbf{m}}}(\partial, \bar{\partial})(h^{-\nu}\phi_{\underline{\lambda}})(0)$ of the product $h^{-\nu}\phi_{\underline{\lambda}}$ of the Bergman reproducing kernel $h^{-\nu}(z, z)$ and the spherical function $\phi_{\underline{\lambda}}$, which is the Clebsch-Gordan coefficient in the tensor product decomposition of a Bergman space with its conjugate and is then the Berezin (integral) transform of the function $K_{\underline{\mathbf{m}}}(z, z)$. As a symmetric functions of $\underline{\lambda}$, $\mathcal{K}_{\underline{\mathbf{m}}}(\partial, \bar{\partial})(h^{-\nu}\phi_{\underline{\lambda}})(0)$ form a system of orthogonal hypergeometric polynomials; see [7]. For rank two domains we can calculate the polynomial for $\underline{\mathbf{m}} = (1, 1)$ by using the result Unterberger and Upmeier [16]; see Proposition 5.2 and Theorem 5.4. We express the operator $\mathcal{M}_3$ and the Shimura operator $\mathcal{M}_{(1,1)}$ in terms of the Rudin type operator $\mathcal{K}_{(1,1)}$ and find their eigenvalues; see Proposition 4.7 and Theorem 6.5.

There is another motivation of our study of the invariant Cauchy-Riemann operator on bounded symmetric domains. When Ω is the unit ball in $\mathbb{C}^n$ and W is a line bundle over Ω we proved earlier [9] a product formula expressing $\mathcal{M}_m$ as a polynomial of $\mathcal{M}_1$. We further proved, by using the product formula, that the powers $\bar{D}^m$ of $\bar{D}$ are intertwining operators realizing the relative discrete series on the line bundle as Bergman spaces of vector-valued functions. We believe that further study of those operators on bounded symmetric domains will help to understand the relative discrete series in line or vector bundles.

Along the way of our study we find also some combinatorial formulas involving invariant theory of the group K. We believe that those formulas are also interesting in their own right.

The paper is organized as follows. In §1 we recall the Jordan triple characterization of a bounded symmetric domain. In §2 we recall and prove some elementary formulas for the invariant polynomials $K_{\underline{\mathbf{m}}}$. Using the polynomials $K_{\underline{\mathbf{m}}}$ we introduce in §3 a system of invariant differential operators $\mathcal{K}_{\underline{\mathbf{m}}}$. We clarify further the relation between $\mathcal{M}_m = D^m\bar{D}^m$ and the Shimura Laplacians $\mathcal{M}_{\underline{\mathbf{m}}}$. In §4 we express $\mathcal{M}_2$ and $\mathcal{M}_3$ in terms of the operators M_1 and $\mathcal{K}_{(1,1)}$. The eigenvalues of $\mathcal{K}_{\underline{\mathbf{m}}}$ are closely related to Berezin transform, and for rank two domains we calculate the eigenvalues of $\mathcal{K}_{(1,1)}$ in §5, thus proving that the two pairs of operators $(\mathcal{K}_{(1,0)}, \mathcal{K}_{(1,1)})$ and (M_1, M_3) form two systems of generators. In §6 we find the relation between the Shimura operator $\mathcal{M}_{(1,1)}$ and the operator $\mathcal{K}_{(1,1)}$ for rank two domains, and thus find its eigenvalues.

We mention here that some of our results in §5–§6 can be generalized easily to line bundles over a general bounded symmetric domain. In particular, it is not difficult to prove that the operators $\mathcal{K}_{\underline{\mathbf{m}}}$, for $\underline{\mathbf{m}}$ being the fundamental representations, are also generators. However, we will not pursue it in the present paper. The eigenvalues of those operators and the Shimura operators bear some remarkable analytical and combinatoric properties. We hope to return to them and some applications of our results in a future paper.

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Notation: We list the main symbols used in this paper.

1. Ω, a Kähler manifold and eventually a bounded symmetric domain G/K in a vector space V;
2. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$;
3. $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k} + \mathfrak{p}^-$, the Harish-Chandra decomposition of $\mathfrak{g}^{\mathbb{C}}$;
4. $P^+K^{\mathbb{C}}P^- \subset G^{\mathbb{C}}$, the Harish-Chandra decomposition of $G^{\mathbb{C}}$;
5. $\mathcal{P}$, the space of holomorphic polynomials on V;
6. $D(z,\bar{w})v = \{z\bar{w}v\}$, the Jordan triple product;
7. $\langle z, w\rangle = \frac{1}{p}\operatorname{Tr} D(z,\bar{w})$, the normalized Hermitian inner product on V;
8. $B(z,w) = I - D(z,\bar{w}) + Q(z)Q(\bar{w})$, the Bergman operator, and $\det B(z,w)^{-1} = h(z,w)^{-p}$ the Bergman reproducing kernel;
9. $\mathcal{D}_G(\Omega)$, the algebra of G-invariant differential operators on Ω;
10. $\bar{D}$, the invariant Cauchy-Riemann operator;

11. $D = -\bar{D}^*$, the formal adjoint of the Cauchy-Riemann operator D;
12. $\mathcal{M}_m = D^m \bar{D}^m$, the invariant Laplacians, and $L = \mathcal{M}_1$, the Laplace-Beltrami operator;
13. $K_{\underline{\mathbf{m}}}(z,z)$, the reproducing kernel of the irreducible subspace of polynomials with signature $\underline{\mathbf{m}}$;
14. $\mathcal{M}_{\underline{\mathbf{m}}}$, the Shimura invariant Laplacians;
15. $\mathcal{K}_{\underline{\mathbf{m}}}$, the invariant Laplacians with symbol $K_{\underline{\mathbf{m}}}$ at $z=0$;
16. $\mathcal{M}(\underline{\boldsymbol{\lambda}})$, the eigenvalue of an invariant differential operator $\mathcal{M}$ on the spherical function $\phi_{\underline{\boldsymbol{\lambda}}}$.

1. Preliminaries

We recall some basic facts about the Jordan triple characterization of bounded symmetric domains; see [6].

Let Ω be an irreducible bounded symmetric domain in a complex n-dimensional space V. Let $G = Aut(\Omega)_0$ be the connected component of the identity in the group $Aut(\Omega)$ of biholomorphic antomorphisms of Ω, and let K be the isotropy subgroup of G at the point 0. Then, as a Hermitian symmetric space, $\Omega = G/K$. Let $G^{\mathbb{C}}$ be the complexification of G as in [6] realized as the automorphism group of the compactification of Ω, and $K^{\mathbb{C}}$ be the Lie subgroup with Lie algebra $\mathfrak{k}$. The Lie algebra $\mathfrak{g}$ of G is identified with the Lie algebra $aut(\Omega)$ of all completely integrable holomorphic vector fields on Ω, equipped with the Lie product

$$[X,Y](z) := X'(z)Y(z) - Y'(z)X(z), \quad X,Y \in aut(\Omega),\ z \in D.$$

Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be the **Cartan decomposition** of $\mathfrak{g}$ with respect to the involution $\theta(X)(z) := -X(-z)$. There exists a quadratic form Q: $V \to End(\overline{V}, V)$ (where $\overline{V}$ is the complex conjugate of V), such that $\mathfrak{p} = \{\xi_v; v \in V\}$, where $\xi_v(z) := v - Q(z)\bar{v}$.

Let $\{z\bar{v}w\}$ be the polarization of the $Q(z)\bar{v}$, i.e.,

$$\{z\bar{v}w\} = Q(z+w)\bar{v} - Q(z)\bar{v} - Q(w)\bar{v}.$$

The space V with the triple product $V \times \overline{V} \times V$ is a JB*-triple; see [18]. Define $D(z,\bar{w}) \in \mathrm{End}(V,V)$ by $D(z,\bar{v})w = \{z\bar{v}w\}$. The space V carries a **K-invariant inner product**

$$\langle z,w\rangle := \frac{1}{p}\mathrm{Tr}D(z,\bar{w}), \tag{1.1}$$

where "Tr" is the trace functional on $End(V)$, and $p = p(\Omega)$ is the genus of Ω (see (1.7) below). We mention the following property of $D(x, \bar{y})$,

$$D(x,\bar{y})^* = D(y,\bar{x}), \tag{1.2}$$

where $D(x, \bar{y})^*$ stands for the Hermitian adjoint of $D(x, \bar{y})$ on V.

Let $B(z, w)$ be the Bergman operator on V,

$$B(z,\bar{w}) = 1 - D(z,\bar{w}) + Q(z)Q(\bar{w}).$$

Identifying the holomorphic tangent space of Ω at a point z with V, the Bergman metric of Ω at z is given by

$$\langle B(z,\bar{z})^{-1}u, v\rangle. \tag{1.3}$$

The Bergman kernel of Ω is then, up to a constant,

$$\det(B(z,w)^{-1}) = h(z,w)^{-p}, \tag{1.4}$$

where $h(z, w)$ is an irreducible polynomial in (z, w). For simplicity we write $h(z) = h(z, z)$.

Let us choose and fix a frame $\{e_j\}_{j=1}^r$ of tripotents in V, where r is the rank of Ω. Then $e := e_1 + \cdots + e_r$ is a **maximal tripotent**. Let

$$V = \sum_{0\le j\le k\le r}^{\oplus} V_{j,k} \tag{1.5}$$

be the **joint Peirce decomposition** of V associated with $\{e_j\}_{j=1}^r$, where

$$V_{j,k} = \{v \in V; D(e_l, e_l)v = (\delta_{l,j} + \delta_{l,k})v,\ 1 \le l \le r\},$$

for $(j, k) \ne (0, 0)$, $V_{0,0} = \{0\}$, and $V_{j,j} = \mathbb{C}e_j$, $1 \le j \le r$. The integers

$$a := \dim V_{j,k}\ (1 \le j < k \le r); \quad b := \dim V_{0,j}\ (1 \le j \le r)$$

are independent of the choice of the frame and of $1 \le j < k \le r$. The Peirce decomposition associated with e is then $V = V_2 \oplus V_1$ with

$$V_2 = \sum_{1\le j\le k\le r} V_{j,k} \quad\text{and}\quad V_1 = \sum_{j=1}^{r} V_{0,j}. \tag{1.6}$$

The **genus** $p = p(\Omega)$ is defined by

$$p := \frac{1}{r}\mathrm{Tr}D(e,\bar{e}) = (r-1)a + b + 2. \tag{1.7}$$

Thus $\langle e_j|e_j\rangle = \frac{1}{p}\mathrm{Tr}D(e_j,\bar{e}_j) = \frac{1}{rp}\mathrm{Tr}D(e,e) = 1$.

Let $\mathfrak{a} = \mathbb{R}\xi_{e_1} + \cdots + \mathbb{R}\xi_{e_r}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ with basis vectors $\xi_{e_1}, \xi_{e_2}, \ldots, \xi_{e_r}$. Let $\{\beta_j\}_{j=1}^r \subset \mathfrak{a}^*$ be the basis of $\mathfrak{a}^*$ determined by

$$\beta_j(\xi_{e_k}) = 2\delta_{j,k}, \quad 1 \le j,k \le r,$$

and define an ordering on $\mathfrak{a}^*$ via

$$\beta_r > \beta_{r-1} > \cdots > \beta_1 > 0.$$

We will write an element $\underline{\lambda} \in (\mathfrak{a}^*)$ as

$$\underline{\lambda} = \sum_{j=1}^r \lambda_j \beta_j,$$

and identify $\underline{\lambda}$ with $(\lambda_1, \lambda_2, \ldots, \lambda_r)$.

The **positive root system** $\sigma^+(\mathfrak{g}, a)$ consists of $\{\beta_j; 1 \le j \le r\}$, $\{(\beta_j \pm \beta_k)/2; 1 \le k < j \le r\}$ and $\{\beta_j/2; 1 \le j \le r\}$, with multiplicities 1, a and $2b$, respectively. It follows that $\underline{\rho}$, the **half sum of the positive roots**, is given by

$$\underline{\rho} = \sum_{j=1}^r \rho_j \beta_j = \sum_{j=1}^r \frac{b+1+a(j-1)}{2}\beta_j. \tag{1.8}$$

2. K-invariant polynomials on bounded symmetric domains

We first recall the decomposition of the polynomial space $\mathcal{P}$ on V under K. It has been done by Hua [5] for classical domains and by Schmid [12] for general domains. See also [2], Theorem 5.4. To state their result we let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$ containing the elements $D(e_j, e_j)$, $j = 1, 2, \ldots, r$. Let $\gamma_1 > \gamma_2 \cdots > \gamma_r$ be the Harish-Chandra strongly orthogonal roots. Thus $\gamma_k(D(e_j, e_j)) = 2\delta_{jk}$. The space $V = \mathfrak{p}^+$ is now of highest weight γ_1 with highest weight vector e_1; and dual space V' is of lowest weight $-\gamma_1$.

THEOREM 2.1 ([5], [12] and [2]): *The space $\mathcal{P}$ of holomorphic polynomials on V decomposes into irreducible subspaces under $Ad(K)$, with multiplicity one as:*

$$\mathcal{P} \cong \sum_{\underline{\mathbf{m}} \ge 0} \mathcal{P}^{\underline{\mathbf{m}}}.$$

Each $\mathcal{P}^{\underline{\mathbf{m}}}$ is of lowest weight $-\underline{\mathbf{m}} = -(m_1\gamma_1 + \cdots + m_r\gamma_r)$ with $m_1 \ge \cdots \ge m_r \ge 0$.

We will hereafter simply call $\underline{\mathbf{m}} = (m_1, m_2, \ldots, m_r)$ (instead of $-\underline{\mathbf{m}}$) the signature of the space $\mathcal{P}^{\underline{\mathbf{m}}}$.

The space $\mathcal{P}$ can be equipped with the Fock norm, defined by

$$(p,q) = p(\partial_z)q^*(z)|_{z=0},$$

where q^* is obtained from q by taking the complex conjugate of the coefficients of q. Let $K_{\underline{\mathbf{m}}}(z,w)$ be the reproducing kernel of $\mathcal{P}^{\underline{\mathbf{m}}}$ with respect to the Fock norm. Thus

$$e^{\langle z,w\rangle} = \sum_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z,w). \tag{2.1}$$

The following expansion of $h(z,w)^{-\nu}$ will be important for our purpose; see [2].

Theorem 2.2 (Faraut and Koranyi [2], Theorem 3.8): *The function $h(z,w)^{-\nu}$ has the following expansion,*

$$h^{-\nu}(z,w) = \sum_{\underline{\mathbf{m}}} (\nu)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z,w) \tag{2.2}$$

for all $\nu \in \mathbb{C}$, and the convergence is uniform on compact subsets of $D \times D$. Here

$$(\nu)_{\underline{\mathbf{m}}} = \prod_{j=1}^{r}\Big(v - \frac{a}{2}(j-1)\Big)_{m_j} = \prod_{j=1}^{r}\prod_{k=1}^{m_j}\Big(v - \frac{a}{2}(j-1) + k - 1\Big).$$

Lemma 2.3: *The following formula holds,*

$$|w|^4 = 2K_{(2,0)}(w,w) + 2K_{(1,1)}(w,w). \tag{2.3}$$

This follows easily by comparing the expansion (2.1) with

$$e^{\langle z,w\rangle} = \sum_{m=0}^{\infty} \frac{\langle z,w\rangle^m}{m!}.$$

Lemma 2.4: *The K-invariant polynomial $\frac{1}{2}\langle D(w,\bar{w})w,w\rangle$ has the following decomposition,*

$$\frac{1}{2}\langle D(w,\bar{w})w,w\rangle = |w|^4 - 2\Big(1+\frac{a}{2}\Big)K_{(1,1)}(w,w).$$

Proof: The group K acts on V as the isomorphism group of the Jordan triple, namely $k\{u\bar{v}w\} = kD(u,\bar{v})w = D(ku,\overline{kv})kw$. Thus $\langle D(w,\bar{w})w,w\rangle$ is K-invariant. The polynomials $|w|^4$ and $K_{(1,1)}(w,w)$ form a basis for the

K-invariant polynomials of degree 4. Thus $\langle D(w,w)w,w\rangle$ can be uniquely written as a linear combination of the two polynomials. To find the coefficients we take $w = s_1e_1 + \cdots + s_re_r$ with $s_j \in \mathbb{R}$. Hence

$$\begin{aligned}\frac{1}{2}\langle D(w,\bar{w})w,w\rangle &= \sum_{j=1}^{r} s_j^4 = (\sum_{j=1}^{r} s_j^2)^2 - 2\sum_{j<k} s_j^2 s_k^2 \\ &= |w|^4 - 2\sum_{j<k} s_j^2 s_k^2. \end{aligned} \tag{2.4}$$

Now, using Theorem 2.2 for $\nu = -1$ we get

$$h(w,w) = \prod_{j=1}^{r}(1-s_j^2) = \sum_{j=0}^{r}(-1)_{\gamma_1+\cdots+\gamma_j} K_{\gamma_1+\cdots+\gamma_j}(w,w)$$

and

$$\sum_{j<k} s_j^2 s_k^2 = (-1)_{(1,1)} K_{(1,1)}(w,w) = \Big(1+\frac{a}{2}\Big) K_{(1,1)}(w,w). \tag{2.5}$$

The lemma follows by substituting (2.5) into (2.4). ∎

LEMMA 2.5: *For any orthonormal basis $\{v_j\}$ of the Jordan triple V we have*

$$\sum_j \operatorname{Tr} D(v_j,\bar{z})D(z,\bar{v}_j) = c|z|^2, \tag{2.6}$$

where

$$c = 2(n+1) + \frac{1}{n}(a^2 - 2^2)\dim \mathcal{P}^{(1,1)} \tag{2.7}$$

and $n = \dim V$.

Proof: By the same reason as in the previous lemma we know that the left hand side of (2.6) is a K-invariant polynomial of z. Being homogeneous of degree 2, it is a constant multiple of $|z|^2$. To evaluate the constant we consider the Euclidean Laplacian operator $\partial\bar{\partial} = \sum_{j=1}^{n} \partial_{v_j}\partial_{\bar{v}_j}$, acting on the function $\operatorname{Tr}(Q(z)Q(\bar{z}))$,

$$\begin{aligned}\partial\bar{\partial}(\operatorname{Tr}(Q(z)Q(\bar{z}))) &= \sum_{j,k} \partial_{v_j}\partial_{\bar{v}_j}\langle Q(z)Q(\bar{z})v_k, v_k\rangle \\ &= \sum_{j,k}\langle Q(z,v_j)Q(\bar{z},\bar{v}_j)v_k, v_k\rangle \\ &= \sum_{j,k}\langle D(z, D(\bar{z},v_k)\bar{v}_j)v_j, v_k\rangle\end{aligned}$$

$$
\begin{aligned}
(2.8)\qquad &= \sum_{j,k}\langle v_j, D(D(z,\bar v_k)v_j,\bar z)v_k\rangle \\
&= \sum_{j,k}\langle v_j, D(v_k,\bar z)D(z,\bar v_k)v_j\rangle \\
&= \sum_{k}\operatorname{Tr}(D(v_k,\bar z)D(z,\bar v_k)) \\
&= c|z|^2.
\end{aligned}
$$

Hence

$$(\partial\bar\partial)^2(\operatorname{Tr}(Q(z)Q(\bar z)) = cn.$$

We calculate the left hand side. The function $\operatorname{Tr}(Q(z)Q(\bar z))$ is K-invariant, and to find the left hand side we need to find its decomposition in terms of basis functions $|z|^4$ and $K_{(1,1)}(z,z)$. Let $z = s_1e_1 + s_2e_2 + \cdots + s_re_r$ be the spectral decomposition of z with $s_j \in \mathbb{R}$. Then by [6], Corollary 3.15,

$$
\begin{aligned}
\operatorname{Tr}(Q(z)Q(\bar z)) &= \sum_{j=1}^{r} s_j^4 + a\sum_{j<k} s_j^2s_k^2 \\
&= \sum_{j=1}^{r} s_j^4 + a\sum_{j<k} s_j^2s_k^2 \\
(2.9)\qquad &= (\sum_{j=1}^{r} s_j^2)^2 + (a-2)\sum_{j<k} s_j^2s_k^2 \\
&= |z|^4 + (a-2)\Big(1+\frac{a}{2}\Big)K_{(1,1)}(z,z).
\end{aligned}
$$

Now

$$(\partial\bar\partial)^2(|z|^4) = 4n + 4\binom{n}{2} = 2n(n+1)$$

by direct calculation; Lemma 2.3 with w replaced by ∂ implies that

$$
\begin{aligned}
(\partial\bar\partial)^2K_{(1,1)}(z,z) &= (2K_{(1,1)}(\partial,\partial) + 2K_{(2,0)}(\partial,\partial))K_{(1,1)}(z,z) \\
(2.10)\qquad &= 2K_{(1,1)}(\partial,\partial)K_{(1,1)}(z,z) = 2\dim\mathcal{P}^{(1,1)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
cn &= 2n(n+1) + 2(a-2)\Big(1+\frac{a}{2}\Big)\dim\mathcal{P}^{(1,1)} \\
&= 2n(n+1) + (a^2-2^2)\dim\mathcal{P}^{(1,1)}.
\end{aligned}
$$

This gives the formula for the constant c. ∎

The constant $\dim\mathcal{P}^{(1,1)}$ is calculated in [17].

3. Invariant Cauchy-Riemann operator $\bar{D}$, Laplacians $\mathcal{M}_m = D^m\bar{D}^m$, the Shimura Laplacians $\mathcal{M}_{\underline{\mathbf{m}}}$ and differential operators $\mathcal{K}_{\underline{\mathbf{m}}}$

The invariant Cauchy-Riemann operator for the unit disk in the complex plane is introduced by Peetre and Zhang in [8], and studied further in [1] and [9].

Let Ω be a Kähler manifold with the Kähler metric locally given by the matrix (h_{ij}) and W a Hermitian vector bundle over Ω. Let $\bar{D}$ be the invariant Cauchy-Riemann operator on E as defined in [1]. Locally, $\bar{D}$ can be defined as follows. Let e_α be a collection of local trivializing sections. If $f = \sum_\alpha f_\alpha e_\alpha$ is any section of W, then

$$\bar{D}f = \sum_{j,i,\alpha} h^{\bar{j}i}\frac{\partial f^\alpha}{\partial \bar{z}^j} v_i \otimes e_\alpha.$$

Here $v_j = \partial_j$ are the basis vectors for the holomorphic tangent space $T_z^{(1,0)} = T_z^{(1,0)}(\Omega)$. Denote $C^\infty(W)$ the space of C^∞-sections of W. Thus

$$\bar{D}\colon C^\infty(W) \mapsto C^\infty(T^{(1,0)} \otimes W),$$

where $T^{(1,0)}$ is the holomorphic tangent bundle over Ω. We recall the following important intertwining property of $\bar{D}$:

$$\bar{D}(f \circ \psi) = \bar{D}f \circ \psi$$

if ψ is any biholomorphic mapping of Ω into itself. The action on sections of the bundles is the induced action. We denote by $\odot^m T^{(1,0)}$ the symmetric tensor subbundle of $\otimes^m T^{(1,0)}$.

The following result is proved in [9].

Lemma 3.1: *Let Ω be a Kähler manifold. Then the iterate $\bar{D}^m$ of the Cauchy-Riemann operator $\bar{D}$ maps $C^\infty(W)$ into $C^\infty\big((\odot^m T^{(1,0)}) \otimes W\big)$.*

We now specialize the above result to an irreducible bounded symmetric domain $\Omega = G/K$ as in §2. Let $W = W_\tau$ be the homogeneous vector bundle over Ω induced by a representation (V^τ, τ) of K. The invariant Hermitian inner product on the sections of the bundle W_τ is given as follows,

$$\int_\Omega \langle \tau(K(z:z))f(z), g(z)\rangle d\iota(z),$$

where $K(z:z)$ is the $K^{\mathbb{C}}$-part of the element $\exp(\bar{z})\exp(z)$ in the Harish-Chandra decomposition of $P^+K^{\mathbb{C}}P^-$ of $G^{\mathbb{C}}$ (see [11]), and

$$d\iota(z) = \frac{dm(z)}{h(z)^p} \tag{3.1}$$

is the invariant (Kähler) measure on Ω, and $dm(z)$ the Lebesgue measure corresponding to the inner product (1.1). We let $D = -\bar{D}^*$ be the adjoint of $\bar{D}$; see [1]. Thus

$$Dg = h(z)^p \sum_j \tau(K(z:z))^{-1} \frac{\partial}{\partial z_j}(h(z)^{-p}\tau(K(z:z))g_j) \tag{3.2}$$

if $g = \sum_j v_j \otimes g_j$ is a function with values in $V \otimes V_\tau$. Here $\{v_j\}$ is an orthonormal basis for V.

The holomorphic tangent space $T_z^{(1,0)}$ can be identified with V. The invariant Cauchy-Riemann operator is, in view of (1.3), given explicitly by

$$\bar{D}f(z) = B(z,z)\bar{\partial}f(z). \tag{3.3}$$

Viewing V as the dual space of $\bar{V}$, namely $V = \bar{V}'$, via the bilinear product (1.1) the above formula amounts to

$$\bar{D}f(\bar{v}) = \langle B(z,z)\bar{\partial}f, v\rangle. \tag{3.4}$$

Similarly $\bar{D}^m f$ can be viewed as a function with value in the dual space $\odot^m\bar{V}$. We will use this identification in the next section.

The symmetric tensor $\odot^m T_z^{(1,0)} = \odot^m V$ is decomposed under K into irreducible spaces with signature $\underline{\mathbf{m}}$, by Theorem 2.1. We let $P_{\underline{\mathbf{m}}}$ be the corresponding orthogonal projections onto the irreducible spaces. We form then the Laplacians

$$\mathcal{M}_m = D^m\bar{D}^m, \qquad \mathcal{M}_{\underline{\mathbf{m}}} = D^m P_{\underline{\mathbf{m}}}\bar{D}^m. \tag{3.5}$$

Thus we have

$$\mathcal{M}_m = \sum_{|\underline{\mathbf{m}}|=m} \mathcal{M}_{\underline{\mathbf{m}}}.$$

Notice that $\mathcal{M}_1 = L$ is the Laplace-Beltrami operator.

In [14] Shimura defined also a family of invariant differential operators on a homogeneous vector bundle over Ω. We will identify our operator $\bar{D}^m$ with the Shimura operator E^m. For that purpose we observe that the operator $\bar{D}^m$ has the following simple formula at $z = 0$,

$$\bar{D}^m f(0) = \sum (\bar{\partial}_{i_m}\cdots\bar{\partial}_{i_1})f(0)v_{i_m}\otimes\cdots\otimes v_{i_1},$$

since any degree of differentiation of $B(z,z)$ with respect to $\bar{\partial}$ vanishes at $z = 0$. Here the summation is over all $(i_1, i_2, \ldots, i_m) \in \{1, 2, \ldots, n\}^m$.

The argument below is quite standard and we will be very brief. Given a V^τ-valued function f on Ω, let F_f be its lift to the group G, defined by

$$F_f(g) = \tau(K(g:0))^{-1}f(g\cdot 0),$$

where $K(g:z)$ is the factor of holomorphy, defined as the $K^{\mathbb{C}}$-part of the element $g\exp(z)$ in the Harish-Chandra decomposition $P^+K^{\mathbb{C}}P^-$ of $G^{\mathbb{C}}$; see [11] for details. The function $F = F_f$ on the group G now transforms according to the character τ of K. The Shimura invariant operator E^m is then defined as follows. For any function F as above E^mF is a $(\odot^m V)\otimes V^\tau = (\odot^m \bar{V}')\otimes V^\tau$-valued function defined by, for any $B_1, B_2, \dots, B_m \in \bar{V}$, viewed as a left-invariant differential operator,

$$E^mF(B_1, B_2, \dots, B_m) = B_1B_2\cdots B_mF.$$

By pulling back the function E^mF to a function on the domain Ω, we get an intertwining operator mapping $C^\infty(W_\tau)$ to $C^\infty((\odot^m T^{(1,0)})\otimes W_\tau)$.

On the other hand, the iterate $\bar{D}^m$ satisfies the same intertwining property as that of E^τ. At the point $z=0$ of Ω, we easily see that the two operators $\bar{D}^m$ and E^m are the same, thus they are the same on Ω. Furthermore, the Shimura Laplacians are defined by $(-1)^m(E^m)^*P_{\underline{\mathbf{m}}}E^m$; see [14], Proposition 4.1 and Theorem 4.3; also [15], formula (2.20). Namely our Laplacian $\mathcal{M}_{\underline{\mathbf{m}}} = ((-\bar{D})^*)^m P_{\underline{\mathbf{m}}}\bar{D}^m = D^mP_{\underline{\mathbf{m}}}\bar{D}^m$ is the same as that of Shimura. Summarizing we obtain

PROPOSITION 3.2: *Realizing as operators acting on V^τ-valued functions on Ω we have $\bar{D}^m = E^m$, $\mathcal{M}_{\underline{\mathbf{m}}} = D^mP_{\underline{\mathbf{m}}}\bar{D}^m = (-1)^m(E^m)^*P_{\underline{\mathbf{m}}}E^m$, and $\mathcal{M}_m = \sum_{|\underline{\mathbf{m}}|}\mathcal{M}_{\underline{\mathbf{m}}}$.*

We consider now the case of the trivial line bundle. We introduce yet another system of invariant differential operators $\mathcal{K}_{\underline{\mathbf{m}}}$ using an idea of Rudin [10], and thus give an explicit linear isomorphism between the algebra $\mathcal{D}_G(\Omega)$ and the algebra of all K-invariant polynomials. The eigenvalues of those operators are simply the coefficients in the expansion of the spherical function $\phi_{\underline{\lambda}}(z)$ in terms of the functions $K_{\underline{\mathbf{m}}}(z,z)$, and are somewhat easier to calculate. Furthermore, they are a special case of a class of orthogonal hypergeometric polynomials in $\underline{\lambda}$; see [7]. We will represent the Laplacians $\mathcal{M}_2, \mathcal{M}_3$ and $\mathcal{M}_{(1,1)}$ in terms of $\mathcal{K}_{\underline{\mathbf{m}}}$ and thus find their eigenvalues.

For each K-invariant polynomial $K_{\underline{\mathbf{m}}}(z,z)$ we associate to it a differential operator $\mathcal{K}_{\underline{\mathbf{m}}}$ by the following,

$$\mathcal{K}_{\underline{\mathbf{m}}}f(z) = K_{\underline{\mathbf{m}}}(\partial,\bar{\partial})f(\psi_z(w))|_{w=0}, \quad z\in\Omega,$$

for any C^∞-function f on Ω, where $\psi_z \in G$ is so that $\psi_z(0) = z$. One easily checks that $\mathcal{K}_{\underline{\mathbf{m}}}$ is G-invariant.

Given a linear differential operator $\mathcal{M} = \sum_{\alpha,\beta} a_{\alpha,\beta}(z)\partial^\alpha\bar{\partial}^\beta$ on Ω with C^∞-coefficients, we let $\sigma_0(\mathcal{M})$ be its symbol at the origin,

$$\sigma_0(\mathcal{M})(w) = \sum_{\alpha,\beta} a_{\alpha,\beta}(0)(\bar{w})^\alpha w^\beta. \tag{3.6}$$

Here we use the usual notation that

$$\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} \quad \text{and} \quad w^\alpha = w_1^{\alpha_1}w_2^{\alpha_2}\cdots w_n^{\alpha_n}$$

in terms of coordinates $(w_1, w_2, \ldots, w_n)$ of w in V. The symbol at the origin can be calculated by the following formula,

$$\sigma_0(\mathcal{M})(w) = \mathcal{M}(e_w)(0), \tag{3.7}$$

where e_w is the function

$$e_w(z) = e^{\langle z,w\rangle + \langle w,z\rangle}. \tag{3.8}$$

In particular, we observe that

$$\sigma_0(\mathcal{K}_{\underline{\mathbf{m}}})(w) = K_{\underline{\mathbf{m}}}(w,w). \tag{3.9}$$

Moreover, for any two invariant differential operators $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M} = \mathcal{N}$ if and only if $\sigma_0(\mathcal{M}) = \sigma_0(\mathcal{N})$. This observation will be used throughout our calculations below.

4. Relations between the operators $\mathcal{M}_m$ and $\mathcal{K}_{\underline{\mathbf{m}}}$

We investigate now the relation between the Laplace operators $\mathcal{M}_m$ and the operators $\mathcal{K}_{\underline{\mathbf{m}}}$. We will express $\mathcal{M}_m$ as a polynomial of $\mathcal{K}_{\underline{\mathbf{m}}}$. As is clear from (3.7) and (3.9), we need to calculate $\sigma(\mathcal{M}_m)(w)$, and express it in terms of $K_{\underline{\mathbf{m}}}(w,w)$.

LEMMA 4.1: *With the definition (3.6) we have*

$$\sigma_0(\mathcal{M}_2)(w) = |w|^4 - 2p|w|^2.$$

Before plunging into the calculation, we remark that the operator D in (3.2) has the following simple form at $z = 0$,

$$Df(0) = \sum_j \partial_j f_j(0), \tag{4.1}$$

if $f = \sum_j v_j \otimes f_j$, with $v_j = \partial_j$. Also, we shall use the usual identification of tensors with linear transformations. For any finite dimensional Hilbert vector space X, the tensor product $X \otimes X$ can be identified with $End(\bar{X}, X)$, the space of linear operators from $\bar{X}$ to X, namely if $T \in End(\bar{X}, X)$ the corresponding tensor is

$$\sum v_j \otimes T\bar{v}_j, \tag{4.2}$$

where $\{v_j\}$ is an orthonormal basis of X. In particular, the operator $Q(z, w) \in End(\bar{V}, V)$ will be identified in this way.

Proof: We calculate $D^2\bar{D}^2$ by definition, using the formula (3.4). Firstly,

$$\bar{D}e_w(z)(\bar{v}_1) = \langle B(z, z)w, v_1\rangle e_w(z).$$

To calculate $\bar{D}^2 e_w$, we note that

$$\begin{aligned} \partial_{\bar{v}} B(z, \bar{z})w &= -\partial_{\bar{v}}(D(z, \bar{z})w) + \partial_{\bar{v}}(Q(z)Q(\bar{z})w) \\ &= -D(z, \bar{v})w + Q(z)Q(\bar{z}, \bar{v})w \\ &= -Q(z, w)\bar{v} + Q(z)D(\bar{z}, w)\bar{v}. \end{aligned} \tag{4.3}$$

Consequently

$$\begin{aligned} \bar{D}^2 e_w(z)(\bar{v}_2, \bar{v}_1) =& e_w(z)(\langle B(z, z)w, v_2\rangle\langle B(z, z)w, v_1\rangle \\ &+ \langle -B(z, z)Q(z, w)\bar{v}_2, v_1\rangle) + \text{Rest} \end{aligned}$$

with the term Rest consisting of terms of homogeneous degree in $(z, \bar{z})$ higher than $(2, 0)$ (namely whose degree in z is higher than 2 or in $\bar{z}$ is higher than 0), which will vanish in $\mathcal{M}_2 e_w(0) = D^2\bar{D}^2 e_w(0)$. Hence, using the formula (4.1),

$$\begin{aligned} \mathcal{M}_2 e_w(0) &= D^2\left(w \otimes w e^{\langle z, w\rangle}\right)\Big|_{z=0} - D^2\left(Q(z, w)e^{\langle z, w\rangle}\right)\Big|_{z=0} \\ &= |w|^4 - 2\,\mathrm{Tr}\, D(w, \bar{w}) = |w|^4 - 2p|w|^2, \end{aligned} \tag{4.4}$$

which is our lemma. ∎

LEMMA 4.2: *We have the following two formulas,*

$$\sigma_0(L^2)(w) = |w|^4 - p|w|^2 \tag{4.5}$$

and

$$\sigma_0(L^3)(w) = |w|^6 - 2\langle D(w, \bar{w})w, w\rangle - 3p|w|^4 + (p^2 + c)|w|^2, \tag{4.6}$$

where the constant c is given in Lemma 2.5.

To simplify the calculation, we note that the Laplace-Beltrami operator $L = \mathcal{M}_1$ has the following form on a Kähler manifold,

$$Lf = \sum_{ij} h^{\bar{j}i} \partial_i \bar{\partial}_j f;$$

see e.g. [1], (3.21).

Proof: Using again the formula (1.3) for the metric $h_{i\bar{j}}$, we get

$$Le_w(z) = e_w(z)\langle B(z,z)w, w\rangle$$

and

$$\begin{aligned} L^2 e_w(z) =& e_w(z)\Big(\langle B(z,z)w,w\rangle^2 \\ &+ \sum_j \langle B(z,z)w, v_j\rangle \Big(-\langle D(v_j,\bar{z})w,w\rangle + \langle Q(v_j,z)Q(\bar{z})w,w\rangle \Big) \\ &+ \langle -B(z,z)Q(z,w)\bar{w}, w\rangle - \sum_j \langle B(z,z)Q(v_j,w)\bar{w}, v_j\rangle \\ &+ \frac{1}{2}\langle B(z,z)D(w,\bar{z})D(z,\bar{w})z, w\rangle \\ &+ \sum_j \langle B(z,z)D(w,\bar{z})D(v_j,\bar{w})z, v_j\rangle \Big) \\ =& e_w(z) f_w(z) \quad \text{(say)}. \end{aligned}$$

Putting $z = 0$ we get the first equality (4.5). Notice again that $L^3 e_w(0) = L(L^2 e_w)(0) = \partial\bar{\partial}(L^2 e_w)(0)$. We only need to find the terms of homogeneous degree $(1,1)$ in z and $\bar{z}$ in the expansion of the $L^2 e_w(z)$ near $z = 0$. In view of the preceding formula,

$$\begin{aligned} f_w(z) =& |w|^4 - p|w|^2 - 2\langle D(z,\bar{z})w,w\rangle - \sum_j \langle w, v_j\rangle \langle D(v_j,\bar{z})w,w\rangle \\ &- \langle Q(z,w)\bar{w},w\rangle - \sum_j \langle D(z,\bar{z})Q(v_j,w)\bar{w}, v_j\rangle \\ &+ \sum_j \langle D(w,\bar{z})D(v_j,\bar{w})z, v_j\rangle + \text{Rest} \end{aligned}$$

where the term Rest is a sum of terms of homogeneous degree higher than $(1,1)$ in z and $\bar{z}$. Similarly

$$e_w(z) = 1 + \langle w,z\rangle + \langle z,w\rangle + \langle w,z\rangle\langle z,w\rangle + \text{Rest}.$$

We find then

$$\begin{aligned} L^3 e_w(0) =& |w|^6 - 2\langle D(w,\bar{w})w, w\rangle - 3|w|^2 \operatorname{Tr} D(w,w) \\ &+ \sum_j \operatorname{Tr}\Big(D(v_j,\bar{v}_j)D(w,\bar{w})\Big) + \sum_j \operatorname{Tr}\Big(D(w,\bar{v}_j)D(v_j,\bar{w})\Big). \end{aligned}$$

The last term is evaluated by Lemma 2.5. To evaluate the second last term, we calculate

$$\begin{aligned} \Big\langle \sum_j D(v_j,\bar{v}_j)x, y\Big\rangle &= \sum_j \langle D(x,\bar{v}_j)v_j, y\rangle \\ &= \sum_j \langle v_j, D(v_j,\bar{x})y\rangle \\ &= \sum_j \langle v_j, D(y,\bar{x})v_j\rangle \\ &= \operatorname{Tr} D(x,\bar{y}) \\ &= p\langle x, y\rangle, \end{aligned}$$

by (1.2) and (1.1). That is $\sum_j D(v_j,\bar{v}_j) = pI$. We get then

$$\sum_j \operatorname{Tr}\Big(D(v_j,\bar{v}_j)D(w,\bar{w})\Big) = p \operatorname{Tr} D(w,\bar{w}) = p^2\langle w, w\rangle.$$

This completes the proof. ∎

Comparing $\sigma_0(L^2)$ and $\sigma_0(\mathcal{M}_2)$ we get

PROPOSITION 4.3: *The following formula holds:*

$$\mathcal{M}_2 = L^2 - pL = L(L-p).$$

Remark 4.4: It has been proved in [1] that generally, when Ω is a Kähler-Einstein manifold, that is when the Ricci tensor is a constant multiple (say k) of the metric tensor, $\mathcal{M}_2 = L(L-k)$. For a bounded symmetric domain Ω, $k = p$, the genus of Ω, our formula thus coincides with theirs. For the tube domain of rank two (i.e., the Lie ball), the above formula is proved in [14], (6.13a).

Observe that

$$\mathcal{M}_2 = DD\bar{D}\bar{D} = D[D,\bar{D}]\bar{D} + D\bar{D}D\bar{D} = D[D,\bar{D}]\bar{D} + L^2.$$

We have the following formula:

COROLLARY 4.5: *The following formula holds:*

$$D[D,\bar{D}]\bar{D} = -pL.$$

It is worth noticing that the above equality can also be written as $D[D,\bar{D}]\bar{D} = D(-p)\bar{D}$; however $[D,\bar{D}]$ (on the holomorphic tangent bundle) is not the constant matrix $-p$.

LEMMA 4.6: *The symbol of the operator $\mathcal{M}_3$ at $z=0$ is given by*

$$\sigma_0(\mathcal{M}_3)(w) = |w|^6 - 3\langle D(w,\bar{w})w,w\rangle - 6p|w|^4 + (6p^2+3c)|w|^2, \tag{4.7}$$

where c is the constant in (2.7).

Proof: We calculate first $\bar{D}^3 e_w$. A straightforward calculation gives

$$\begin{aligned}
&\bar{D}^3 e_w(z)(\bar{v}_1,\bar{v}_2,\bar{v}_3)\\
=&e_w(z)\Big(\langle w,v_3\rangle\langle w\otimes w - D(z,\bar{z})w\otimes w - w\otimes D(z,\bar{z})w, v_2\otimes v_1\rangle\\
&\quad - \langle Q(z,w)\bar{v}_3,v_2\rangle\langle w,v_1\rangle - \langle w,v_2\rangle\langle Q(z,w)\bar{v}_3,v_1\rangle\\
&\quad - \langle w,v_3\rangle\langle Q(z,w)\bar{v}_2,v_1\rangle + \langle D(z,\bar{v}_3)Q(z,w)\bar{v}_2,v_1\rangle\\
&\quad + \langle Q(z)D(\bar{v}_3,w)\bar{v}_2,v_1\rangle\Big) + \text{Rest}.
\end{aligned}$$

Here the term Rest, as before, consists of terms that will vanish in $\mathcal{M}_3 e_w(0)$. We find now

$$\begin{aligned}
\mathcal{M}_3 e_w(0) =&|w|^6 - 3\langle D(w,w)w,\bar{w}\rangle - 6p|w|^4 + 6p^2|w|^2\\
&+3\sum_{j=1}^{n}\mathrm{Tr}D(v_j,\bar{w})D(w,\bar{v}_j)|w|^2.
\end{aligned}$$

The last term is then evaluated in Lemma 2.5. ∎

We have now a formula expressing $\mathcal{M}_3$ as a polynomial of L and $\mathcal{K}_{(1,1)}$.

PROPOSITION 4.7: *The operator $\mathcal{M}_3$ is a polynomial of $L=\mathcal{M}_1$ and $\mathcal{K}_{(1,1)}$. More precisely*

$$\mathcal{M}_3 = L^3 - (3p+2)L^2 + (2p^2-2p+2c)L + 2^2\Big(1+\frac{a}{2}\Big)\mathcal{K}_{(1,1)}.$$

Proof: The formulas (4.6) and (4.7) imply that

$$\sigma(\mathcal{M}_3 - L^3)(w)$$

$$
\begin{aligned}
&= -\langle D(w,\bar{w})w, w\rangle - 3p|w|^4 + 5p^2|w|^2 + 2c|w|^2 \\
&= -2|w|^4 + 2^2\left(1+\frac{a}{2}\right)K_{(1,1)}(w,w) - 3p|w|^4 + 5p^2|w|^2 + 2c|w|^2 \\
&= -(3p+2)(|w|^4 - p|w|^2) + (2p^2 - 2p + 2c)|w|^2 + 2^2\left(1+\frac{a}{2}\right)K_{(1,1)}(w,w) \\
&= -(3p+2)\sigma_0(L^2)(w) + (2p^2 - 2p + 2c)|w|^2 + 2^2\left(1+\frac{a}{2}\right)K_{(1,1)}(w,w),
\end{aligned}
$$

where we use Lemma 2.4 in the second equality and (4.5) in the last one. This proves the proposition. ∎

Remark 4.8: When Ω is the unit ball in $\mathbb{C}^n$, the term $\mathcal{K}_{(1,1)}$ will not appear, and $p = n+1$, $c = 2(n+1) = 2p$. Thus

$$\mathcal{M}_3 = L^3 - (3p+2)L^2 + (2p^2 + 2p)L = L(L-p)(L - 2(p+1)).$$

A general product formula for $\mathcal{M}_m$ has been proved in [1], and for $\mathcal{M}_m$ on line bundles in [9].

5. Eigenvalues of the operators $\mathcal{K}_{\underline{\mathbf{m}}}$ and $\mathcal{M}_m$

We let $\phi_{\underline{\lambda}}(z)$ be the spherical function on Ω as defined in [3], Chapter IV. Let $L = D\bar{D}$ be the Laplace-Beltrami operator on Ω. We have $L = \mathcal{M}_{(1,0)}$ and, for $\underline{\lambda} = \sum_{j=1}^r \lambda_j \beta_j$, $\phi_{\underline{\lambda}}$ is a eigenfunction of L with eigenvalue

$$-\frac{1}{4}((\underline{\lambda},\underline{\lambda}) + (\underline{\rho},\underline{\rho})) = -\sum_{j=1}^r \lambda_j^2 - \sum_{j=1}^r \rho_j^2 \tag{5.1}$$

since β_j has norm 2. (Here the factor $\frac{1}{4}$ appears because of the usual convention with the complex differentiation, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ for $z = x + iy$.)

In this section we will calculate the eigenvalues of the invariant differential operator $\mathcal{K}_{(1,1)}$. For this purpose we will use the the Berezin transform. The application of Berezin transform to invariant differential operators has been studied in our early papers [9], [7]. We notice that the eigenvalues $\mathcal{K}_{\underline{\mathbf{m}}}(\underline{\lambda})$ of $\mathcal{K}_{\underline{\mathbf{m}}}$ on the spherical functions are given by

$$\mathcal{K}_{\underline{\mathbf{m}}}(\underline{\lambda}) = (\mathcal{K}_{\underline{\mathbf{m}}}\phi_{\underline{\lambda}})(0). \tag{5.2}$$

Our idea is that, instead of calculating (5.2), we calculate

$$\mathcal{K}_{\underline{\mathbf{m}}}(h(z,z)^{-\nu}\phi_{\underline{\lambda}}(z))|_{z=0},$$

which, roughly speaking, is an integral of the spherical function $\phi_{\underline{\lambda}}$ against the function $K_{\underline{\mathbf{m}}}(z,z)$ and can be evaluated for certain $\underline{\mathbf{m}}$ using the Berezin transform.

For $\nu > p-1$ the Berezin transform B_ν is defined by

$$B_\nu f(z) = c_\nu \int_D \frac{h(z,z)^\nu h(w,w)^\nu}{h(z,w)^\nu h(w,z)^\nu} f(w) d\iota(w),$$

where $d\iota(w)$ is the invariant measure (3.1), and

$$c_\nu := \Big(\int_D h(z,z)^\nu d\iota(z)\Big)^{-1} = \frac{1}{\pi^n} \frac{\prod_{j=1}^r \Gamma(\nu - \frac{a}{2}(j-1))}{\prod_{j=1}^r \Gamma(\nu - \frac{n}{r} - \frac{a}{2}(j-1))}$$

is the normalizing constant. The operator B_ν defines an invariant bounded self-adjoint operator on $L^2(\Omega)$ with the invariant measure $d\iota(z)$; see [16]. The spectral symbol $b_\nu(\underline{\lambda})$ of B_ν is given by the eigenvalue of B_ν on $\phi_{\underline{\lambda}}$. More exactly, Unterberger and Upmeier [16] proved that the integral

$$B_\nu \phi_{\underline{\lambda}}(z) = c_\nu \int_D \frac{h(z,z)^\nu h(w,w)^\nu}{h(z,w)^\nu h(w,z)^\nu} \phi_{\underline{\lambda}}(w) d\iota(w) \tag{5.3}$$

is absolutely convergent for $\underline{\lambda}$ in an open domain of $(\mathfrak{a}^*)^{\mathbb{C}}$ and equals $b_\nu(\underline{\lambda})\phi_{\underline{\lambda}}(z)$, and calculated the symbol $b_\nu(\underline{\lambda})$.

THEOREM 5.1 (Unterberger and Upmeier [16], Proposition 3.39): *The spectral symbol of the Berezin transform is given by*

$$b_\nu(\underline{\lambda}) = \prod_{j=1}^r \frac{\Gamma(i\lambda_j + \nu - \frac{p-1}{2})\Gamma(-i\lambda_j + \nu - \frac{p-1}{2})}{\Gamma(\rho_j + \nu - \frac{p-1}{2})\Gamma(-\rho_j + \nu - \frac{p-1}{2})}.$$

Unless otherwise mentioned we will assume from now on that Ω is an irreducible bounded symmetric domain of rank two.

PROPOSITION 5.2: *The following differentiation formula holds:*

$$\mathcal{K}_{(1,1)}(h^{-\nu}\phi_{\underline{\lambda}})(0) = \frac{1}{(-1)_{(1,1)}}\Big\{\Big(\big(\nu - \frac{p-1}{2}\big)^2 + \lambda_1^2\Big)\Big(\big(\nu - \frac{p-1}{2}\big)^2 + \lambda_2^2\Big) - (\nu)_{(1,1)}^2 + (\nu - \frac{a}{2})^2\Big[n\nu - \frac{1}{4}(\underline{\lambda},\underline{\lambda}) - \frac{1}{4}(\underline{\rho},\underline{\rho})\Big]\Big\}.$$

Proof: We take $\underline{\lambda}$ in an open domain so that the integral (5.3) is absolutely convergent. (Such an open domain exists; see [16].) Notice that the equality $b_\nu(\underline{\lambda})\phi_{\underline{\lambda}}(z) = B_\nu\phi_{\underline{\lambda}}(z)$ can also be written as

$$b_\nu(\underline{\lambda}) h(z)^{-\nu}\phi_{\underline{\lambda}}(z) = c_\nu \int_D \frac{h(w,w)^\nu}{h(z,w)^\nu h(w,z)^\nu} \phi_{\underline{\lambda}}(w) d\iota(w).$$

Let the operator $\mathcal{K}_{\underline{\mathbf{m}}}(\partial, \bar{\partial})$ act on the equality at $z = 0$; we get

$$\begin{aligned} & b_\nu(\underline{\boldsymbol{\lambda}})\mathcal{K}_{\underline{\mathbf{m}}}\left(h^{-\nu}\phi_{\underline{\boldsymbol{\lambda}}}\right)(0) \\ =& b_\nu(\underline{\boldsymbol{\lambda}})K_{\underline{\mathbf{m}}}(\partial, \bar{\partial})\left(h^{-\nu}\phi_{\underline{\boldsymbol{\lambda}}}\right)(0) \qquad (5.4)\\ =& c_\nu \int_D K_{\underline{\mathbf{m}}}(\partial_z, \bar{\partial}_z)\left(\frac{1}{h(z,w)^\nu h(w,z)^\nu}\right)(0) h(w,w)^\nu \phi_{\underline{\boldsymbol{\lambda}}}(w)d\iota(w); \end{aligned}$$

using Theorem 2.2 we get

$$K_{\underline{\mathbf{m}}}(\partial_z, \bar{\partial}_z)\left(\frac{1}{h(z,w)^\nu h(w,z)^\nu}\right)(0) = (\nu)_{\underline{\mathbf{m}}}^2 K_{\underline{\mathbf{m}}}(w,w).$$

Therefore the right hand side of the preceding formula (5.4) can be written as the Berezin transform of $K_{\underline{\mathbf{m}}}(w,w)\phi_{\underline{\boldsymbol{\lambda}}}(w)$ at $z = 0$, namely,

$$c_\nu(\nu)_{\underline{\mathbf{m}}}^2 \int_D K_{\underline{\mathbf{m}}}(w,w)\phi_{\underline{\boldsymbol{\lambda}}}(w)d\iota(w) = (\nu)_{\underline{\mathbf{m}}}^2 B_\nu(K_{\underline{\mathbf{m}}}\phi_{\underline{\boldsymbol{\lambda}}})(0),$$

or

$$b_\nu(\underline{\boldsymbol{\lambda}})K_{\underline{\mathbf{m}}}(\partial, \bar{\partial})\left(h^{-\nu}\phi_{\underline{\boldsymbol{\lambda}}}\right)(0) = (\nu)_{\underline{\mathbf{m}}}^2 B_\nu(K_{\underline{\mathbf{m}}}\phi_{\underline{\boldsymbol{\lambda}}})(0). \qquad (5.5)$$

Let first $\underline{\mathbf{m}} = (1,0)$. Then $K_{(1,0)}(\partial, \bar{\partial}) = \sum_{j=1}^n \partial_j\bar{\partial}_j$ is the Euclidean Laplacian.. We calculate the right hand side in (5.5). First we notice that

$$\partial_j h(z)^{-\nu}(0) = \bar{\partial}_j h(z,z)^{-\nu}(0) = 0$$

and

$$K_{(1,0)}(\partial, \bar{\partial})(h^{-\nu})(0) = n(\nu)_{(1,0)}$$

by the expansion of $h(z,z)^{-\nu}$ in Theorem 2.2. Thus

$$\begin{aligned} K_{(1,0)}(\partial, \bar{\partial})(h^{-\nu}\phi_{\underline{\boldsymbol{\lambda}}})(0) &= K_{(1,0)}(\partial, \bar{\partial})(h^{-\nu})(0) + K_{(1,0)}(\partial, \bar{\partial})\phi_{\underline{\boldsymbol{\lambda}}}(0) \\ &= n(\nu)_{(1,0)} + \mathcal{K}_{(1,0)}(\underline{\boldsymbol{\lambda}}). \end{aligned}$$

The second term is the eigenvalue of the Laplacian operator, namely $-\frac{1}{4}((\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}}) + (\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{\rho}}))$. The formula (5.5) can be rewritten as

$$\nu^2 B_\nu(K_{(1,0)}\phi_{\underline{\boldsymbol{\lambda}}})(0) = b_\nu(\underline{\boldsymbol{\lambda}})\left(n\nu - \frac{1}{4}((\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}}) + (\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{\rho}}))\right). \qquad (5.6)$$

We now calculate the right hand side in (5.5) for $\underline{\mathbf{m}} = (1,1)$. The expansion (2.2) for $\nu = -1$ implies

$$h(w,w) = 1 - K_{(1,0)}(w,w) + (-1)_{(1,1)}K_{(1,1)}(w,w),$$

or

$$(-1)_{(1,1)}K_{(1,1)}(w,w) = h(w,w) - 1 + K_{(1,0)}(w,w).$$

Therefore

$$(-1)_{(1,1)}B_\nu(K_{(1,1)}\phi_{\underline{\lambda}})(0) = B_\nu(h\phi_{\underline{\lambda}})(0) - B_\nu(\phi_{\underline{\lambda}})(0) + B_\nu(K_{(1,0)}\phi_{\underline{\lambda}})(0).$$

The first term can be evaluated by Theorem 5.1. Indeed,

$$\begin{aligned} B_\nu(h\phi_{\underline{\lambda}})(0) &= \frac{c_\nu}{c_{\nu+1}}B_{\nu+1}(\phi_{\underline{\lambda}})(0) = \frac{c_\nu}{c_{\nu+1}}b_{\nu+1}(\underline{\lambda}) \\ &= \frac{c_\nu}{c_{\nu+1}}b_\nu(\underline{\lambda})\frac{\prod_{j=1}^2((\nu-\frac{p-1}{2})^2+\lambda_j^2)}{\prod_{j=1}^2((\nu-\frac{p-1}{2})^2-\rho_j^2)}; \end{aligned} \tag{5.7}$$

and the second term $B_\nu(\phi_{\underline{\lambda}})(0) = b_\nu(\underline{\lambda})$, again by Theorem 5.1. The third term is evaluated by (5.6). Summing up we have obtained

$$\begin{aligned} &b_\nu(\underline{\lambda})\mathcal{K}_{(1,1)}\left(\phi_{\underline{\lambda}}h^{-\nu}\right)(0) \\ =&(\nu)^2_{(1,1)}b_\nu(\underline{\lambda})\frac{1}{(-1)_{(1,1)}}\left\{\frac{c_\nu}{c_{\nu+1}}\frac{\prod_{j=1}^2((\nu-\frac{p-1}{2})^2+\lambda_j^2)}{\prod_{j=1}^2((\nu-\frac{p-1}{2})^2-\rho_j^2)} - 1\right. \\ &\left.+\frac{1}{\nu^2}\left[n\nu - \frac{1}{4}((\underline{\lambda},\underline{\lambda}) + (\underline{\rho},\underline{\rho}))\right]\right\}. \end{aligned}$$

Multiplying both sides by $b_\nu(\underline{\lambda})^{-1}$, we get

$$\begin{aligned} \mathcal{K}_{(1,1)}\left(h^{-\nu}\phi_{\underline{\lambda}}\right)(0) =&\frac{(\nu)^2_{(1,1)}}{(-1)_{(1,1)}}\left\{\frac{c_\nu}{c_{\nu+1}}\frac{\prod_{j=1}^2((\nu-\frac{p-1}{2})^2+\lambda_j^2)}{\prod_{j=1}^2((\nu-\frac{p-1}{2})^2+\rho_j^2)} - 1\right. \\ &\left.+\frac{1}{\nu^2}\left[n\nu - \frac{1}{4}(\underline{\lambda},\underline{\lambda}) - \frac{1}{4}(\underline{\rho},\underline{\rho})\right]\right\}, \end{aligned}$$

which, after simplifying, is our result. ∎

Remark 5.3: The polynomial in the right hand side of the above proposition is one of a system of hypergeometric polynomials in $\underline{\lambda}$ orthogonal with respect to the measure $|c(\underline{\lambda})|^{-2}b_\nu(\underline{\lambda})$; here $c(\underline{\lambda})$ is the Harish-Chandra c-function; see [7].

The right hand side in the above Proposition is a polynomial in ν. Taking $\nu = 0$ we have now

THEOREM 5.4: *The eigenvalue of the differential operator $\mathcal{K}_{(1,1)}$ is*

$$\begin{aligned} &\mathcal{K}_{(1,1)}(\underline{\lambda}) \\ =&\frac{1}{1+\frac{1}{2}a}\left(-\left(\frac{a}{2}\right)^2(\lambda_1^2+\lambda_2^2+\rho_1^2+\rho_2^2) + \left(\left(\frac{p-1}{2}\right)^2+\lambda_1^2\right)\left(\left(\frac{p-1}{2}\right)^2+\lambda_2^2\right)\right). \end{aligned}$$

We notice that the symmetric polynomials $\mathcal{K}_{(1,1)}(\underline{\lambda})$ and $\mathcal{K}_{(1,0)}(\underline{\lambda})$ generate the elementary symmetric polynomials $\lambda_1^2 + \lambda_2^2$ and $\lambda_1^2\lambda_2^2$. We get now immediately

COROLLARY 5.5: *The operators $\mathcal{K}_{(1,0)}$ and $\mathcal{K}_{(1,1)}$ form a system of generators for the algebra $\mathcal{D}_G(\Omega)$.*

From Proposition 4.7 one can calculate the eigenvalue of $\mathcal{M}_3$. Notice that the operator $\mathcal{K}_{(1,1)}$ appears in $\mathcal{M}_3$ linearly. We have then the following result, which proves a conjecture of Englis and Peetre [1] for rank two domains.

THEOREM 5.6: *The operators $\mathcal{M}_1$ and $\mathcal{M}_3$ form a system of algebraically independent generators of the algebra $\mathcal{D}_G(\Omega)$.*

The above theorem can be proved somewhat more easily, without the exact calculation in Proposition 4.7 and Theorem 5.4. We observe that $\mathcal{M}_3$ and L^3 have the same leading term. Thus we need only to find the 4-th order term in $\sigma_0(\mathcal{M}_3) - \sigma_0(L^3)$ and prove that it can be written as $\alpha|w|^4 + \beta K_{(1,1)}(w,w)$ for some non-zero constant β. We remark further that, among the bounded symmetric domains, there are two exceptional domains with rank 2 and 3 for which the algebra $\mathcal{D}_G(\Omega)$ is larger than the image of the center of the universal enveloping algebra [4]. The above theorem thus gives a geometric construction of generators of the algebra $\mathcal{D}_G(\Omega)$. It would be interesting to understand if the conjecture is true for rank 3 domains.

6. The eigenvalue of the Shimura operator $\mathcal{M}_{(1,1)}$

We calculate now the eigenvalues of the Shimura operator $\mathcal{M}_{(1,1)} = D^2P_{(1,1)}\bar{D}^2$ by expressing it in terms of the operator $\mathcal{K}_{(1,1)}$.

PROPOSITION 6.1: *The symbol of the operator $\mathcal{M}_{(1,1)}$ is*

$$\sigma_0(\mathcal{M}_{(1,1)}) = 2K_{(1,1)}(w,w) + \frac{2(1+b)a}{2+a}|w|^2.$$

Proof: It follows from the proof of Lemma 4.1 that

$$(6.1)\quad \sigma_0(\mathcal{M}_{(1,1)})(w) = D^2P_{(1,1)}(w\otimes we^{\langle z,w\rangle})|_{z=0} - D^2P_{(1,1)}(Q(z,w)e^{\langle z,w\rangle})|_{z=0}.$$

Clearly, by K-invariance

$$D^2P_{(1,1)}(w\otimes we^{\langle z,w\rangle})|_{z=0} = C_1K_{(1,1)}(w,w)$$

for some constant C_1. To find the constant, we observe that similarly

$$D^2P_{(2,0)}(w\otimes we^{\langle z,w\rangle})|_{z=0} = C_2K_{(2,0)}(w,w);$$

using $D^2 = D^2 P_{(1,1)} + D^2 P_{(2,0)}$ we get

$$|w|^4 = D^2\big(w \otimes w e^{\langle z,w\rangle}\big)|_{z=0} = C_1 K_{(1,1)}(w,w) + C_2 K_{(2,0)}(w,w).$$

However, $K_{\underline{\mathbf{m}}}$ form a basis for the K-invariant polynomials; by Lemma 2.3 we get $C_1 = C_2 = 2$. To find the second term in (6.1) we need the following

LEMMA 6.2: *The following differentiation formula holds:*

$$D^2 P_{(1,1)}\big(Q(z,w)e^{\langle z,w\rangle}\big)|_{z=0} = -\frac{2(1+b)a}{2+a}|w|^2. \tag{6.2}$$

Accepting temporarily the lemma we get then the Proposition. ∎

We now prove the lemma.

Proof: Clearly

$$D^2 P_{(1,1)}\big(Q(z,w)e^{\langle z,w\rangle}\big)\Big|_{z=0} = C|w|^2 \tag{6.3}$$

for some constant C. To find the constant C we let $w = e_1$ be a minimal tripotent. Thus $|w|^2 = 1$. We first calculate $P_{(1,1)}Q(z,e_1)$. Let $V = V_2(e_1) \oplus V_1(e_1) \oplus V_0(e_1)$ be the Peirce decomposition, with $V_2(e_1) = \mathbb{C}e_1$. We observe that $P_{(1,1)}e_1 \otimes e_1 = 0$ since $e_1 \otimes e_1$ is a highest weight vector of weight $2\gamma_1 = (2\ 0)$. By the Peirce product rule $\{V_i(e_1)\overline{V_j(e_1)}V_k(e_1)\} \subset V_{i-j+k}$ we have, as a tensor in $V \otimes V$,

$$Q(e_1, e_1) = 2e_1 \otimes e_1 + \sum_j w_j \otimes w_j$$

where $\{w_j\}$ is an orthonormal basis of $V_1(e_1)$, each of which can be chosen as a minimal tripotent. $P_{(1,1)}e_1 \otimes e_1 = 0$ implies $P_{(1,1)}w_j \otimes w_j = 0$ since K acts transitively on minimal tripotents and $P_{(1,1)}$ is K-equivariant. Therefore $P_{(1,1)}Q(z,e_1) = 0$ if $z \in V_2(e_1)$. Now if $z \in V_1(e_1)$ the tensor $Q(z,e_1)$ is of weight

$$\gamma_1 + \frac{\gamma_1+\gamma_2}{2} = \gamma_1 + \gamma_2 + \frac{\gamma_1 - \gamma_2}{2}$$

which is higher than $\gamma_1 + \gamma_2 = (1,1)$, thus again $P_{(1,1)}Q(z,e_1) = 0$. We only need to consider $z \in V_0(e_1)$.

We start to calculate $P_{(1,1)}Q(z,e_1)$. Choose an orthonormal basis $v_1, \ldots, v_{1+b}$ of $V_0(e_1)$ (orthogonal with respect to the Hermitian metric $\langle\cdot,\cdot\rangle$, not as an orthogonal frame of the Jordan triple V_0) consisting of minimal tripotents. We have

$$Q(z,e_1) = \sum_{j=1}^{1+b}\langle z, v_j\rangle Q(v_j, e_1),$$

and the constant C in (6.3) is given by
(6.4)

$$D^2 P_{(1,1)}\left(Q(z,e_1)e^{\langle z,e_1\rangle}\right)\Big|_{z=0} = \sum_{j=1}^{1+b} D^2\left(e^{\langle z,e_1\rangle}\langle z,v_j\rangle P_{(1,1)}Q(v_j,e_1)\right)\Big|_{z=0}.$$

We fix $v_j = e_2$ a minimal tripotent in $V_0(e_1)$. We claim that

$$D^2\Big(e^{\langle z,e_1\rangle}\langle z,e_2\rangle P_{(1,1)}Q(e_2,e_1)\Big)\Big|_{z=0} = -\frac{2a}{2+a}. \tag{6.5}$$

Thus the constant C is $(1+b)\frac{2a}{2+a}$, which is our lemma. We now prove (6.5). We will find $P_{(1,1)}Q(e_2,e_1)$ in terms of the highest weight vector in the subspace with signature $(1,1)$ of the symmetric tensor $\odot^2 V$. Let

$$V = (V_{11}\oplus V_{22}\oplus V_{12})\oplus(V_{01}\oplus V_{02})$$

with $V_{ij} = V_{ji}$, be the joint Peirce decomposition of V. Consequently we get the Peirce decomposition with respect to $e = e_1 + e_2$,

$$V = V_2(e)\oplus V_1,\quad V_2(e) = V_{11}\oplus V_{22}\oplus V_{12},\quad V_1(e) = V_{01}\oplus V_{02}.$$

Using the Peirce product rule

$$\{V_{ij}\bar{V_{jk}}V_{kl}\}\subset V_{il}$$

and that all other triple products are 0, we get $Q(e_2,e_1)\in V_2(e)\odot V_2(e)$. Now $V_2(e)$ is a Jordan algebra of rank 2 and thus is equivalent to the Jordan triple of type IV; see [6], 4.11. We can thus identify $V_2(e)$ with the type IV Jordan triple $\mathbb{C}^{2+a}$ and the triple product on $\mathbb{C}^{2+a}$ is given by

$$D(z,\bar{v})u = Q(z,u)\bar{v} = (z\cdot\bar{v})u + (u\cdot\bar{v})z - 2(z\cdot u)\bar{v}.$$

Here $z\cdot u$ is the quadratic form $\sum_j z_j u_j$. We can assume that $e_1 = \frac{1}{2}(1,i)$ and $e_2 = \frac{1}{2}(1,-i)$. Let $u_j, j = 1,2,\ldots,a$ be an orthonormal basis of V_{12}. Thus $Q(e_2,e_1)\bar{u_j} = -u_j$ and

$$Q(e_2,e_1) = -\sum_{j=1}^{a} u_j\otimes u_j.$$

The vectors u_j are all of weight $\frac{1}{2}(\gamma_1+\gamma_2)$, thus $P_{(1,1)}Q(e_1,e_2)$ is a constant multiple of the highest weight vector

$$q = e_1\otimes e_2 + e_2\otimes e_1 + \sum_{j=1}^{a} u_j\otimes u_j$$

in the subspace with signature $(1,1)$. By elementary calculation we find the decomposition

$$\sum_{j=1}^{a} u_j \otimes u_j = \frac{a}{2+a}q + \left(\frac{2}{2+a}\sum_{j=1}^{a} u_j \otimes u_j - \frac{a}{2+a}(e_1 \otimes e_2 + e_2 \otimes e_1)\right),$$

where the second vector is orthogonal to q, namely is in $(2,0)$-space. Hence

$$P_{(1,1)}Q(e_2, e_1) = -\frac{a}{2+a}q.$$

The left hand side of (6.5) is now

$$\begin{aligned} D^2\big(e^{\langle z,e_1\rangle}\langle z, e_2\rangle P_{(1,1)}Q(e_2,e_1)\big)|_{z=0} &= -\frac{a}{2+a}D^2\big(e^{\langle z,e_1\rangle}\langle z,e_2\rangle q)\big)|_{z=0} \\ &= -\frac{2a}{2+a}. \end{aligned}$$

This proves (6.5) and thus the lemma. ∎

We obtain consequently a formula expressing the Shimura operator $\mathcal{M}_{(1,1)}$ in terms of $\mathcal{K}_{(1,0)}$.

PROPOSITION 6.3: *The following formula holds,*

$$\mathcal{M}_{(1,1)} = 2\mathcal{K}_{(1,1)} + \frac{2a(1+b)}{a+2}\mathcal{K}_{(1,0)}$$

with $\mathcal{K}_{(1,0)} = L$.

Remark 6.4: Recall Proposition 4.7. We can further write $\mathcal{M}_3$ as a polynomial of the operator L and $\mathcal{M}_{(1,1)}$, in view of the above Proposition:

$$\mathcal{M}_3 = L^3 - (3p+2)L^2 + (2p^2 - 2p + 2c - 2a(1+b))L + (2+a)\mathcal{M}_{(1,1)}.$$

Consider the case of a tube domain or rank two, i.e. the Lie ball. The above formula can be rewritten as

$$\mathcal{M}_3 = L(L - (2n+2))(L-n) + nM_{(1,1)}.$$

This coincides with Shimura [14] (see the third formula in Proposition 6.4 there).

Using Theorem 5.4 we further find the eigenvalues of the Shimura operator.

THEOREM 6.5: *The eigenvalue of the Shimura Laplacian $\mathcal{M}_{(1,1)}$ is*

$$\begin{aligned}
&\mathcal{M}_{(1,1)}(\underline{\lambda})\\
=&\frac{2}{1+\frac{a}{2}}\left((\frac{a}{2})^2(-\lambda_1^2-\lambda_2^2-\rho_1^2-\rho_2^2)+((\frac{p-1}{2})^2+\lambda_1^2)((\frac{p-1}{2})^2+\lambda_2^2)\right)\\
&+\frac{2a(1+b)}{a+2}((-\lambda_1^2-\lambda_2^2-\rho_1^2-\rho_2^2)\\
=&\frac{(1+2\,b+b^2+4\,\lambda_1^2)\,(1+2\,b+b^2+4\,\lambda_2^2)}{4\,(2+a)}.
\end{aligned}$$

Note that the eigenvalue of $\mathcal{M}_{(1,1)}(\underline{\lambda})$, after all, has a nice product formula. In a subsequent paper we will give a different proof of the formula.

We specialize our result for a certain special value of $\underline{\lambda}$. Recall Siegel domain realization of G/K. Let $V = V_2(e) \oplus V_1(e)$ be the Peirce decomposition of V with respect to the maximal tripotent e as in (1.6). The domain $\Omega = G/K$ can also be realized as

$$\{(z_2, z_1) \in V_2(e) \oplus V_1(e); \Im(z_2) - \frac{1}{2}\{z_1\bar{z}_1 e\} > 0\}.$$

Here $\Im z_2$ stands for the imaginary part of $z_2 \in V_2$ in a splitting of V_2, and $y > 0$ when y is an element in the cone of positive elements in a real Jordan algebra; see [6], §10 for the precise formulation. Let $\det(x)$ be the determinant function on the Jordan algebra $V_2(e)$. The function $\det(\Im(z_2) - \frac{1}{2}\{z_1\bar{z}_1e\})^s$, for $s \in \mathbb{C}$, is then an eigenfunction of the algebra $\mathcal{D}_G(\Omega)$. In fact, it is the Harish-Chandra $e_{\underline{\lambda}}$-function

$$\det(\Im(z_2) - \frac{1}{2}\{z_1\bar{z}_1 e\})^s = e_{\underline{\lambda}}(z)$$

with

$$\underline{\lambda} = -i(s\beta_1 + s\beta_2) - \underline{\rho}.$$

See [16]. Substituting the $\underline{\lambda}$ into Theorem 6.5, we get now

COROLLARY 6.6: *When $\Omega = G/K$ is realized as a Siegel domain the function $\det(\Im z_2 - \{z_1\bar{z}_1 e\})^s$ with $s \in \mathbb{C}$ is an eigenfunction of the Shimura Laplacian $\mathcal{M}_{(1,1)}$ with eigenvalue*

$$\begin{aligned}
&-\frac{(a-2s)(2+a+2b-2s)(1+b-s)s}{2+a}\\
=&\frac{4}{a+2}s\left(s-\frac{a}{2}\right)\left(s-\frac{1+b}{2}\right)\left(s-1-b-\frac{a}{2}\right).
\end{aligned}$$

When Ω is a tube domain of complex and symmetric complex square matrix domains, the above result is proved by Shimura in [13], Proposition 11.12. Note that our $\mathcal{M}_{(1,1)}$ differs by a constant from that of Shimura there, as we here are using a different normalization. (The coefficient of the highest degree of our differential operator at the origin is, according to Proposition 6.1 and formula (2.5),

$$2\frac{1}{1+\frac{a}{2}} = \frac{4}{a+2},$$

thus the coefficient in the above formula, whereas the operator in [13] (see formula (11.21b)) has 1 as its coefficient of the highest degree.)

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